## Electromagnetic Waves

In my experience people entering a spectroscopic field from a non-physics background (such as chemistry or biology) can struggle with some of the concepts surrounding the mathematical description of electromagnetic waves, as I did early in my PhD. For a trained physicist, this is generally just assumed knowledge and so it can be demoralising for a new student to not understand what is meant by terms like 'phase', or 'k-vector'. However, it is rather straightforward if broken down simply. A more mathematical treatment can be found in a standard text on electrodynamics, such as [1].

A one-dimensional travelling electromagnetic wave $E(z,t)$ can be expressed as follows:

$$E(z,t) = E_0 \exp\left[\mathrm{i}(kz - \omega t)\right] \tag{1}$$

Physically, this wave could be the electric field of some laser light. The wave is 'travelling' because it is moving in both space (the coordinate $z$), and time $t$. Let us consider the meaning of each of the terms in Equation 1 in turn.

- $E_0$ refers to the **amplitude** of the electric field - the maximum height of the peaks in the wave. This would have units of volts per unit length.
- $t$ is time, with units of time.
- $\omega$ is the **angular frequency** at which the wave oscillates *in time*. It has units of radians per unit time, such that the product $\omega t$ has units of radians, which are dimensionless[1]. This is normally just called the **frequency**. *The more cycles the wave completes per unit time, the larger the frequency.*
- $z$ is the position of the wave in space along the $z$-axis, with units of length.
- $k$ is the **angular wavenumber** of the wave. This has units of radians per unit length, such that the product $kz$ has units of (dimensionless) radians. It is normally just known as the **wavenumber**. This can be thought of as a **spatial frequency**, where $\omega$ was a **temporal frequency**. *The more cycles the wave completes per unit length, the larger the wavenumber.*
- $\mathrm{i}$ is the imaginary unit, defined such that $\mathrm{i}^2 = -1$. This will be discussed further below.

[1] A radian is defined as the ratio of arc length to radius length of a circle, thus the units of length cancel out and the radian is dimensionless.

The wave is written in an exponential form, but is really just a sinusoidal wave, as we know from Euler's formula that:

$$\exp(\mathrm{i}\theta) = \cos(\theta) + \mathrm{i}\sin(\theta) \tag{2}$$

So if we took the real part of our wave $E(z,t)$:

$$\mathrm{Re}[E(z,t)] = E_0 \cos(kz - \omega t) \tag{3}$$

Which is the sinusoidal form we expect. We use this exponential form as it is the most general, and it makes manipulating the wave much simpler when we try to add phase factors and things. To visualise all the parameters above, it is easiest to plot the wave, but before we do this there is some mathematical complexity that needs to be cleared up.

You will probably encounter multiple definitions of the wavenumber, unfortunately. Within spectroscopy and chemistry it is normally thought of as the reciprocal of the vacuum wavelength of a particular spectroscopic transition, with the symbol $\tilde{\nu}$:

$$\tilde{\nu} = \frac{1}{\lambda} \tag{4}$$

This definition is useful in chemistry where desire is really just to have a number that is linked to the transition wavelength but is directly proportional to transition energy. However, in the context of laser physics, we define the wavenumber, $k$, as:

$$k = \frac{2\pi}{\lambda} \tag{5}$$

Which has units of radians per unit length, as mentioned above. This is really the 'angular wavenumber', but it is normally just called the 'wavenumber', like the angular frequency is just called the frequency. We do this because we are always talking about waves that are periodic, and what is interesting is how often the wave completes a complete periodic 'revolution' around $2\pi$ radians (a circle). So how many radians our oscillating wave moves through in a propagation length, or propagation time, is what interests us.

With this in mind, we should think about how we can link together *the number of oscillations per unit length* ($k$) with *the number of oscillations per unit time* ($\omega$). It seems natural that these should be connected: if the wave is oscillating through a certain number of radians in a certain time, and is also moving through space, then the **speed at which it moves through space** will dictate how many radians it oscillates through in a given distance. That is:

$$k = \frac{\omega}{v_{\mathrm{p}}} \tag{6}$$

Where $v_{\mathrm{p}}$ is the **phase velocity** of the wave, which is how fast it is moving in whatever medium it is travelling in. If the wave moves faster, $v_{\mathrm{p}}$ is larger, and the wave won't have time to oscillate through as many radians in a given distance than it would if it was moving more slowly. This is simply what Equation 6 expresses mathematically.

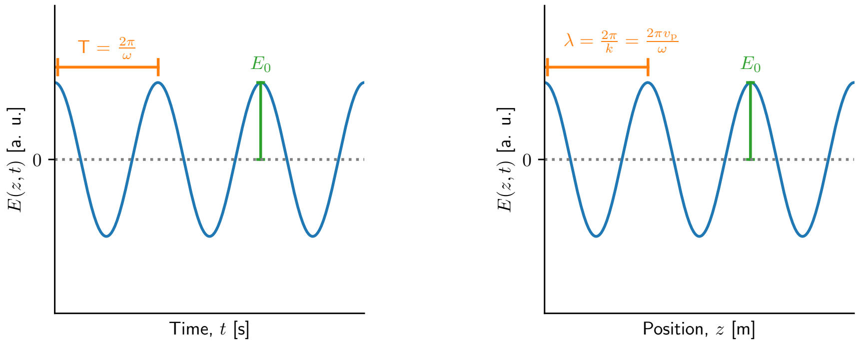


Figure 0.1: Illustration of our wave $E(z,t)$ plotted as a function of time (left) and position (right). Quantities discussed in the text are annotated.

Figure 0.1 shows graphically all of the quantities we have discussed, and how they relate to one another. We have also defined the reciprocal of the frequency, the **oscillation period**, $T = {}^{2\pi}/_{\omega}$, in the leftmost plot on Figure 0.1; and the **wavelength**, $\lambda$, in the rightmost plot. Both of these quantities represent the time taken ($T$) for, and the distance travelled ($\lambda$) in one full oscillation (through $2\pi$ radians). It is clear from the figure that the wavelength, $\lambda$, can be defined in terms of previously met quantities as:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v_{\mathrm{p}}}{\omega} \tag{7}$$

However, at the start of this section we specifically said that we would clarify the meaning of the **phase** and of the **wave vector** - two of the concepts that cause most confusion in this topic in my experience of teaching it. Before we discuss the phase in the following section, we will briefly discuss the wave vector, $\boldsymbol{k}$.

## Wave Vectors

The wave vector $\boldsymbol{k}$ looks like the wave number $k$, but is in bold. This is because the wave vector is a **vector**, so has a direction and a magnitude. **The magnitude of the wave vector is simply the wave number**. The wave vector points in the direction of propagation of our electromagnetic wave (the direction of propagation of a laser beam, for example). In the case that the wave is travelling in 3D space, rather than the 1D case shown above, then the direction of travel can be split into three components, corresponding to movement along the $x, y$ and $z$ axes. In this case we make the substitution:

$$kz \to \boldsymbol{k} \cdot \boldsymbol{r} = k_{\mathrm{x}}\hat{\boldsymbol{x}} + k_{\mathrm{y}}\hat{\boldsymbol{y}} + k_{\mathrm{z}}\hat{\boldsymbol{z}} \tag{8}$$

Where $\boldsymbol{r}$ is our position vector in 3D space, which can be split into three components in terms of the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$. The magnitude of each of these components is $k_\mathrm{x}, k_\mathrm{y}$ and $k_\mathrm{z}$ respectively. The total magnitude $k_\mathrm{3D}$ of our 3D wave $\boldsymbol{k}$ is given straightforwardly by:

$$|\boldsymbol{k}| = k_\mathrm{3D} = \sqrt{k_\mathrm{x}^2 + k_\mathrm{y}^2 + k_\mathrm{z}^2} \tag{9}$$

The value $k_\mathrm{3D}$ given in Equation 9 is simply the three-dimensional analogue of the one-dimensional $k$ used Equation 1. To summarise, the important points are:

- The wave vector, $\boldsymbol{k}$, is a vector that points in the direction that the wave is propagating in.
- The magnitude of the wave vector is the wave number, $k$, and tells you how many cycles the wave completes in a unit propagation distance (units of $\mathrm{m}^{-1}$).

In the context of lasers, we often talk about **k-vectors**, and these are just another name for wave vectors. Specifically, we talk about them in the context of momentum conservation in phase-matching in non-linear optics. This is because we can link the magnitude of the k-vector $k$ to the momentum of the wave $p$:

$$p = \hbar k \tag{10}$$

A simple dimensional analysis illustrates this. Momentum has units of $\mathrm{kg\,m\,s^{-1}}$. The magnitude of the k-vector has units of $\mathrm{rad\,m^{-1}}$ (as discussed above). The reduced Planck's constant $\hbar$ has units of $\mathrm{kg\,m^2\,s^{-2}\,rad^{-1}}$. So, a larger k-vector (or shorter wavelength, or higher frequency) corresponds to a higher momentum for the wave. We will now turn to a discussion of **the phase**.

## Phase

The concept of phase elicits a lot of confusion among students in my experience, but it need not. Fundamentally, **the phase of a wave tells you which part of the cycle it is in.** For example, a wave with a phase of $\pi$ rad is halfway through a cycle, and a wave with a phase of 0 rad is at the beginning of a cycle. As such, **the phase is an angle**, given in radians. Recall however that a radian is dimensionless, so you will equally see it said that the phase is dimensionless. The phase is generally given between 0 and $2\pi$, but sometimes the phase can be greater than $2\pi$, if a wave finishes a cycle and goes onto the next one[2].

[2] This lies at the heart of the concept of **phase unwrapping**. If you are looking at the phase of a signal over a long time, then you might see lots of jumps as the phase gets to $2\pi$ and then skips back to zero. Unwrapping the phase gets rid of these jumps and gives you a continuous phase.

So the phase is just an angle that tells you which part of the oscillation period you are in. If it seems odd that we use an angle to define this, remember that our wave is a periodic sinusoidal function, as shown in Equation 3. The argument that this function takes is an angle, so it's natural that we use an angle to define where 'on' this function we are. However, we can distinguish between the **absolute**

**phase** of the wave, and a **phase shift** or **accumulated phase** that is added to the wave.

The **absolute phase** tells us exactly where we are in the wave cycle overall. To find the absolute phase of our one-dimensional travelling wave, we note that we wrote Equation 1 in exponential form deliberately. This exponential form may be familiar from study of complex numbers in mathematics, where a complex number $N$ can be written as:

$$N = |N| \exp(\mathrm{i}\Theta) \tag{11}$$

Where $|N|$ is the modulus (magnitude) of the complex number, and $\Theta$ is the argument (phase) of the complex number. So we can identify the stuff in the exponent next to the imaginary unit as our phase. This means that the absolute phase of the wave in Equation 1, which we will call $\Theta$, is given by:

$$\Theta = kz - \omega t \tag{12}$$

Both quantities $kz$ and $\omega t$ are angles as discussed above, so the difference $kz - \omega t$ is also an angle, and defines the absolute phase of our wave. You may sometimes see it written as:

$$\Theta = kz - \omega t + \arg(E_0) \tag{13}$$

Which accounts for the possibility that the amplitude $E_0$ is also a complex number with it's own phase. The argument of the complex number is just the phase, so $\arg(E_0)$ is the phase of the amplitude which also contributes to the total absolute phase. But in the examples we consider here $E_0$ is simply a number, so does not have a phase (or has a phase of zero).

We can further exploit the beauty of the exponential form of complex numbers to understand what we mean by a **phase shift** or **accumulated phase**. A phase shift is when we move our wave along in its cycle by a given angle. For illustration, we will call this angle $\phi$. Ultimately what we are doing with a phase shift is:

$$\Theta' = \Theta + \phi \tag{14}$$

Where $\Theta'$ is the absolute phase of the wave after the shift, $\Theta$ is the absolute phase of the wave before the shift, and $\phi$ is the added phase. The exponential form of complex numbers makes this trivially simple. To shift our initial travelling wave $E(z,t)$ by $\phi$ radians, we simply do:

$$E(z,t) \times \exp(\mathrm{i}\phi) = E_0 \exp\left[\mathrm{i}(kz - \omega t + \phi)\right] \tag{15}$$

So multiplication by the **phase factor** $\mathrm{e}^{\mathrm{i}\phi}$ caused a phase shift of our wave by $\phi$ radians. This is what we mean when we talk about phase shifts or accumulated phase, here we have accumulated a phase of $\phi$ radians.

We can get a feel for what this looks like by considering what happens if we shift our wave by some different values of $\phi$. This is illustrated in Figure 0.2. Here we still plot our waves as a function of time, but you can see that the added phase has the effect of moving the part of the cycle that a wave is in at a *given time* around. That is, if you were to pick a specific point on the time axis (such as that shown with the dashed line on the rightmost plot in Figure 0.2), then all the waves are clearly at different points in their cycle. The orange curve is $\pi/2$ radians ahead of the blue curve, and so on. This kind of plot can be difficult to visualise as it looks as though the waves with a positive phase shift are being moved backwards. The best way to think about it is to look at the unshifted wave (blue), and then look at where it is at the dashed line. Now consider where the blue wave would be in $\pi/2$ radians time - this is where the orange wave is at the dashed line.

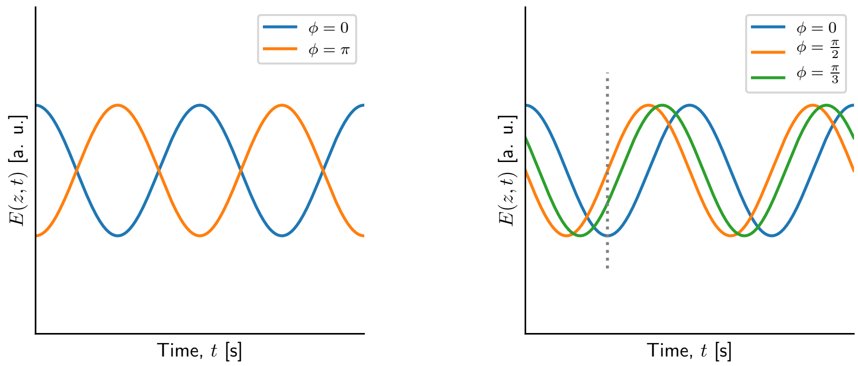


Figure 0.2: Illustration of our wave $E(z, t)$ plotted as a function of time with different phase shifts, $\phi$, applied.

To end with a laser based example, often we talk about the spectral phase $\phi(\omega)$ that is accumulated by a pulse as it propagates through a medium. This just means that a wave in the pulse with frequency $\omega$ is moved in its cycle by $\phi(\omega)$ on propagation through the medium. $\phi(\omega)$ could be a complicated function of $\omega$, which is what gives rise to the GDD and higher order dispersions that cause our pulse to broaden during the propagation.

# Bibliography

[1] Griffiths D J. *Introduction to Electrodynamics*. Cambridge University Press, Cambridge, 4th edition, 2017.